Differential Equation Based Method for Accurate Approximations in Optimization

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This paper describes a method to efficiently and accurately approximate the effect of design changes on structural response. The key to this new method is to interpret sensitivity equations as differential equations that may be solved explicitly for closed-form approximations, hence, the method is denoted the differential equation based (DEB) method. Approximations were developed for vibration frequencies, mode shapes, and static displacements. The DEB approximation method was applied to a cantilever beam and results compared with the commonly used linear Taylor series approximations and exact solutions. The test calculations involved perturbing the height, width, cross-sectional area, tip mass, and bending inertia of the beam. The DEB method proved to be very accurate, and in most cases, was more accurate than the linear Taylor series approximation. The method is applicable to simultaneous perturbation of several design variables. Also, the approximations may be used to calculate other system response quantities. For example, the approximations for displacements are used to approximate bending stresses.

Nomenclature

K = stiffness matrix
M = mass matrix
s = number of steps
v = design variable
v = vector of design variables
Δv = change in design variable
θ = distance along move direction
Φ = vibration eigenvector (mode shape)
ω^2 = vibration eigenvalue (frequency)

Subscripts

i = ith component of vector
o = nominal value

Superscript

T = transpose

Introduction

IT is highly desirable in optimization to be able to estimate the effect of design changes on system behavior without having to perform expensive finite element analyses inside an optimization loop. This need has led to an increased interest in the development of accurate and efficient approximation techniques. Presently, many optimization procedures use linear Taylor series approximations of the objective function and constraints so that excessive calls to the full analysis can be eliminated.[1] Taylor series and other approximations[2] are also used outside of optimization for quick assessment of the effect of design changes. Other techniques and demonstra-

tions of their benefits are described in Refs. 3–11. In this paper, an efficient method is described that significantly extends the range of applicability of approximations beyond the range of the linear Taylor series approach. The key to this new approach is to recognize that the formulas for the sensitivity derivatives of system response quantities can be interpreted as differential equations that may be solved to obtain closed-form approximations. Herein, these approximations will be referred to as the differential equation based (DEB) approximations.

Basis of the Differential Equation Based Method

In this section of the paper, the DEB method is developed for approximating the effect of design changes on response. First, the method will be developed for the approximation of frequencies, then mode shapes, and finally for static displacements.

Approximation of Frequencies

From Ref. 12, the equation for the derivative of a non-repeated vibration eigenvalue ω^2 with respect to a design variable v is

$$\frac{d\omega^2}{dv} = \Phi^T \left[\frac{dK}{dv} - \omega^2 \frac{dM}{dv} \right] \Phi \qquad (1)$$

where Φ is the eigenvector normalized such that $\Phi^T M \Phi = 1$. Define two scalar quantities a and b as follows:

$$a = \Phi^T \frac{dM}{dv} \Phi, \qquad b = \Phi^T \frac{dK}{dv} \Phi \qquad (2)$$

then Eq. (1) may be written as

$$\frac{d\omega^2}{dv} = b - a\omega^2 \qquad (3)$$

Equation (3) may be interpreted as a first-order differential equation in ω^2 with variable coefficients. If a and b do not vary excessively with v, they may be evaluated at the nominal design and considered to be constant. Then the general solution to Eq. (3), provided a is nonzero is

$$\omega^2 = ce^{-av} + (b/a) \qquad (4)$$

Solving for c from the condition of $\omega^2 = \omega_o^2$ when $v = v_o$ and substituting into Eq. (4) gives an explicit approximation for ω^2 as

$$\omega^2 = [\omega_o^2 - (b/a)]e^{-a(v-v_o)} + (b/a) \tag{5}$$

Approximation of Mode Shapes

From Ref. 13, the equation for the derivative of the mode shape Φ with respect to a design variable is

$$\frac{d\Phi}{dv} = Q + D\Phi \tag{6}$$

where Q satisfies the equation

$$[K - \omega^2 M]Q = \frac{d\omega^2}{dv} M\Phi - \frac{dK}{dv}\Phi + \omega^2 \frac{dM}{dv}\Phi \tag{7}$$

and

$$D = -\Phi^T M Q - \frac{1}{2}\Phi^T \frac{dM}{dv}\Phi \tag{8}$$

Derivations of Q and D may be found in Ref. 13. Equation (6) is a nonlinear first-order vector differential equation in Φ with variable coefficients. In order to solve this equation, Q and D are evaluated at the nominal design and assumed constant. Equations (1) and (5) may be used for explicit forms of $d\omega^2/dv$ and ω^2 in the calculations of Q and D. The solution to Eq. (6) is

$$\Phi = Ae^{Dv} - (1/D)Q \tag{9}$$

where A is determined from the nominal condition that $\Phi = \Phi_o$ when $v = v_o$. Solving for A and substituting into Eq. (9) gives the explicit approximation for Φ as

$$\Phi = [\Phi_o + (1/D)Q]e^{D(v-v_o)} - (1/D)Q \tag{10}$$

Equation (10) is a vector equation, however, it is uncoupled in the sense that each component of Φ varies independently with the design variable v. The difference between components is reflected in the corresponding components of Φ and Q.

Approximation of Displacements

Assuming a linear elastic structure in which the applied force does not vary with v, the equation for the displacement derivative is

$$\frac{dU}{dv} = BU \tag{11}$$

where

$$B = -K^{-1}\frac{dK}{dv} \tag{12}$$

Equation (11) is a differential equation that can be solved explicitly for an approximation of the displacement U. Similar to solving the differential equations for frequencies and mode shapes, the matrix B is evaluated at the nominal design and assumed constant for moderate excursions from the nominal design. Equation (11) is a vector differential equation having the following solution

$$U = e^{Bv}A \tag{13}$$

where e^B is the matrix exponential defined as

$$e^B \equiv I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots \tag{14}$$

Solving for A in Eq. (13) from the condition that $U = U_o$ at $v = v_o$ gives

$$U = e^{B(v-v_o)}U_o \tag{15}$$

Using Eq. (14) in Eq. (15) gives the displacement approximation

$$U = U_o + B(v - v_o)U_o + \frac{B^2(v - v_o)^2}{2!}U_o + \cdots \tag{16}$$

Rather than compute the matrix B and its powers and then multiply by U_o, the $B^n U_o$ terms were obtained iteratively as follows:

$$KBU_o = -\frac{dK}{dv}U_o \tag{17a}$$

$$KB^2U_o = -\frac{dK}{dv}BU_o \tag{17b}$$

$$\vdots$$

$$KB^nU_o = -\frac{dK}{dv}B^{n-1}U_o \tag{17c}$$

The Taylor series expansion of U is

$$U = U_o + \frac{dU}{dv}(v - v_o) + \frac{1}{2!}\frac{d^2U}{dv^2}(v - v_o)^2 + \cdots \tag{18}$$

As pointed out in Ref. 14, when K is a linear function of v, Eq. (18) can be written as

$$U = U_o + BU_o(v - v_o) + B^2U_o(v - v_o)^2 + \cdots \tag{19}$$

Also pointed out in Ref. 14, Eqs. (16) and (19) differ only by the factorial in each term. Equation (16) is less accurate than the Taylor series equation (19) when they both converge. Equation (16) is based on the power series for the exponential and will converge for any value of Δv, whereas the Taylor series may not always converge. Since the terms in both Eqs. (16) and (19) are easily evaluated using Eqs. (17), the DEB method would not be the preferred approximation unless the Taylor series did not converge.

Differential Equation Based Method Extended to Multiple Design Variables

The approximations in Eqs. (5), (10), and (15) are directly applicable only when a single design variable is perturbed. To calculate the effect of simultaneous changes in several design variables, the DEB method is modified as follows. If v_o is the vector of nominal design variables and v is the vector of perturbed design variables, then $v - v_o$ is the direction of the move from the nominal to the perturbed design. Any design along the move direction may be associated with a scalar θ between 0 and 1. Then

$$v(\theta) = v_o + \theta(v - v_o) \tag{20}$$

In particular, $v(1) = v$ and $v(0) = v_o$. Now the sensitivity equations (1), (6), and (11) may each be written with θ as the independent variable. For example, Eq. (1) is rewritten as

$$\frac{d\omega^2}{d\theta} = \Phi^T\left[\frac{dK}{d\theta} - \omega^2\frac{dM}{d\theta}\right]\Phi \tag{21a}$$

or

$$\frac{d\omega^2}{d\theta} = b_\theta - a_\theta\omega^2 \tag{21b}$$

where

$$a_\theta = \Phi^T \frac{\mathrm{d}M}{\mathrm{d}\theta} \Phi, \qquad b_\theta = \Phi^T \frac{\mathrm{d}K}{\mathrm{d}\theta} \Phi \qquad (22)$$

and

$$\frac{\mathrm{d}M}{\mathrm{d}\theta} = \sum_{i=1}^{n} \frac{\mathrm{d}M}{\mathrm{d}v_i} \frac{\mathrm{d}v_i}{\mathrm{d}\theta} = \sum_{i=1}^{n} \frac{\mathrm{d}M}{\mathrm{d}v_i} (v_i - v_{oi}) \qquad (23a)$$

$$\frac{\mathrm{d}K}{\mathrm{d}\theta} = \sum_{i=1}^{n} \frac{\mathrm{d}K}{\mathrm{d}v_i} \frac{\mathrm{d}v_i}{\mathrm{d}\theta} = \sum_{i=1}^{n} \frac{\mathrm{d}K}{\mathrm{d}v_i} (v_i - v_{oi}) \qquad (23b)$$

where n is the number of design variables. Now the solution to Eq. (21b) is

$$\omega^2(\theta) = \left(\omega_o^2 - \frac{b_\theta}{a_\theta} \right) e^{-a_\theta(\theta - \theta_o)} + \frac{b_\theta}{a_\theta} \qquad (24)$$

Using Eq. (20), $\omega^2(1) = \omega^2(v)$ and by definition $\theta_o = 0$ and $\theta = 1$ at the perturbed design. Therefore, the final form of Eq. (24) is

$$\omega^2(v) = \left(\omega_o^2 - \frac{b_\theta}{a_\theta} \right) e^{-a_\theta} + \frac{b_\theta}{a_\theta} \qquad (25)$$

Equation (25) corresponds to one step $|v - v_o|$. For a step size of $s|v - v_o|$

$$\omega^2(v) = \left(\omega_o^2 - \frac{b_\theta}{a_\theta} \right) e^{-a_\theta s} + \frac{b_\theta}{a_\theta} \qquad (26)$$

Extension of the method to multiple design variables has also been carried out for Φ and U.

Differential Equation Based Method Extended to Nonlinear Design Variables

We will define a nonlinear design variable to be one that contributes nonlinearly to either the mass or stiffness matrix, and a linear variable is one that contributes linearly to both the mass and stiffness matrices. An early attempt at applying the DEB method in a straightforward manner to nonlinear design variables proved to be highly inaccurate. In retrospect, the reason for the inaccuracy was that the coefficients that were assumed constant (e.g., a and b) were, in fact, highly nonlinear functions of the design variable. The solution to this difficulty is as follows. Define y to be a vector of nonlinear design variables and v to be a vector of linear variables. It is assumed that the linear variables may be written as functions of the nonlinear design variables such that $v_o = v(y_o)$ and $v = v(y)$. For example, the bending inertia I and the cross-sectional area A contribute linearly to M and K and they are both functions of the nonlinear variable H. Then, Eq. (20) may be rewritten with v as a function of y and θ

$$v(\theta,y) = v_o + \theta(v - v_o) = v(y_o) + \theta[v(y) - v(y_o)] \qquad (27)$$

The method for nonlinear design variables then becomes very similar to the DEB method for multiple linear variables [Eq. (26)]. For this case

$$\frac{\mathrm{d}M}{\mathrm{d}\theta} = \sum_{i=1}^{n} \frac{\mathrm{d}M}{\mathrm{d}v_i} \frac{\mathrm{d}v_i}{\mathrm{d}\theta} = \sum_{i=1}^{n} \frac{\mathrm{d}M}{\mathrm{d}v_i} [v_i(y) - v_i(y_o)] \qquad (28a)$$

$$\frac{\mathrm{d}K}{\mathrm{d}\theta} = \sum_{i=1}^{n} \frac{\mathrm{d}K}{\mathrm{d}v_i} \frac{\mathrm{d}v_i}{\mathrm{d}\theta} = \sum_{i=1}^{n} \frac{\mathrm{d}K}{\mathrm{d}v_i} [v_i(y) - v_i(y_o)] \qquad (28b)$$

Then a_θ and b_θ are calculated from Eqs. (22). Unlike the method for linear variables, a_θ and b_θ are evaluated at each step along the move direction due to the nonlinearity of the perturbed variable, and in Eq. (26), s is always equal to 1. Evaluating a_θ and b_θ involves a recalculation of $v_i(y)$ for each

Table 1　Material properties and cross-sectional dimensions of beam model

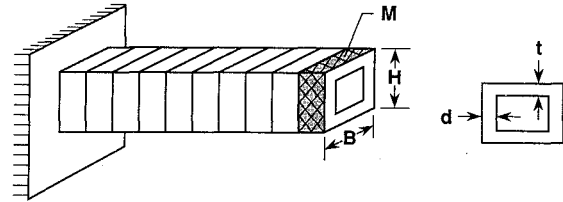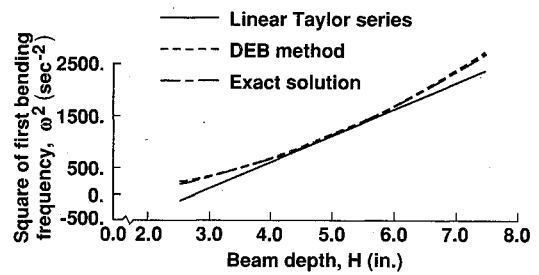| Element number	E, psi	ρ, lb/in.^{3}	H, in.	B, in.	t, in.	d, in.
1	4.90E6	0.07	5.00	3.75	0.80	0.10
2–10	5.85E6	0.07	5.00	3.75	0.80	0.10

Fig. 1　Test problem.

Fig. 2　Comparison of DEB and Taylor series approximations of frequency for perturbation of beam depth H ($B = 3.75$ in.; $M = 0$. lbm).

perturbed step of y and the multiplication of a matrix by a scalar. The derivatives of the mass and stiffness matrices do not have to be recalculated. They are evaluated once at the nominal design and used throughout the range of the move direction. Again, a similar extension has been carried out for Φ and U.

Results

The approximations for frequencies, mode shapes, and displacements were tested on a 193-in. cantilever beam modeled with 10 equal elements (see Fig. 1). The beam has a box cross section and a tip mass. Properties of the beam are listed in Table 1. The test cases involved perturbing the height H, width B, cross-sectional area A, tip mass M, and bending inertia I of the beam. All calculations were generated in a general purpose finite element program (EAL).[15] Results were compared with exact solutions and the linear Taylor series approximations.

Frequency Approximation Results

The DEB method shows excellent capability for approximating frequencies of perturbed designs. For example, Fig. 2 shows a graph of the fundamental eigenvalue vs the height of the beam H for values perturbed from the nominal value of 5.0 in. For as much as a 50% increase in H, the new approximation is within 2% of the exact solution compared to 12% for the linear Taylor series approximation. It is also evident from the figure that, beyond a 45% decrease in H, the linear Taylor series method gives negative values for the eigenvalues. This is an example of the DEB method applied to a nonlinear variable. Figure 3 shows a comparison between the approximations for the frequency due to a change in the magnitude of the tip mass. Again, the new approximation is very accurate. Only a 4% deviation from the exact solution is seen for a 100% increase in the tip mass compared to a 10% deviation for the linear Taylor series approximation. Figure 4 illustrates the application of the DEB method to simultaneous changes in three design variables: tip mass, bending inertia, and cross-sectional area of the beam. Each step s represents a 10%
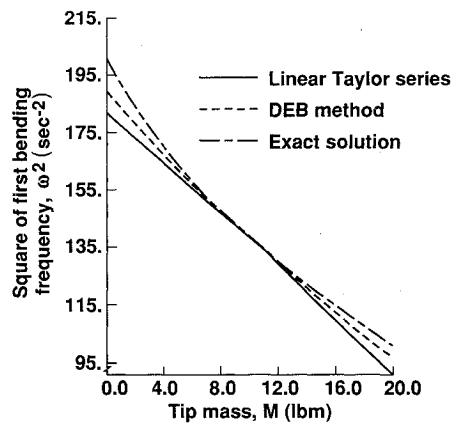
Fig. 3   Comparison of DEB and Taylor series approximations of frequency for perturbation of tip mass ($B = 3.75$ in.; $H = 5.0$ in.).
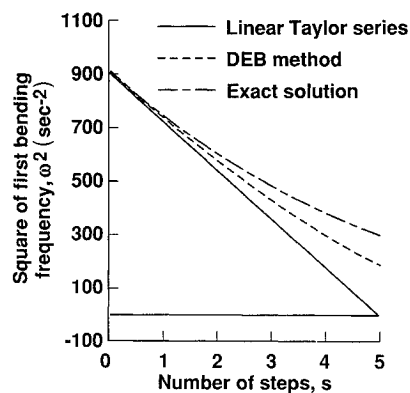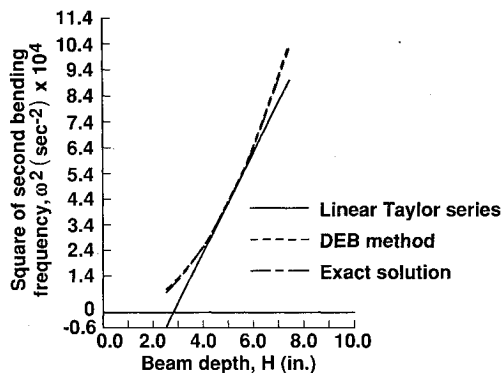

Fig. 4   Comparison of DEB and Taylor series approximation of frequency for simultaneous perburbation of tip mass M, bending inertia I, and cross-sectional area A (nominal values: $M = 5.0$ lbm; $I = 28.863$ in.^{4}; $A = 7.0$ in.^{2}).

Fig. 5   Comparison of DEB and Taylor series approximations of frequency for perturbation of beam depth H($B = 3.75$ in.; $M = 0$. lbm).

perturbation in the design variables from their nominal values. In comparing the DEB approximation with the exact solution, there was a 36% error for 50% increase in the design variables. The linear Taylor series approximation was in error by nearly 100%. Figure 5 shows the DEB and linear Taylor series approximations of the second bending frequency compared to the exact solution for perturbed values of the beam depth H. Again, the new method is very accurate. For as much as a 50% increase in H, the DEB approximation is within 2% of the exact solution compared to 12% for the linear Taylor series. This result is very similar to the result for the first bending mode with H as the design variable. Similar trends occurred in perturbation studies of the other variables for approximating frequencies of the first and second bending modes.
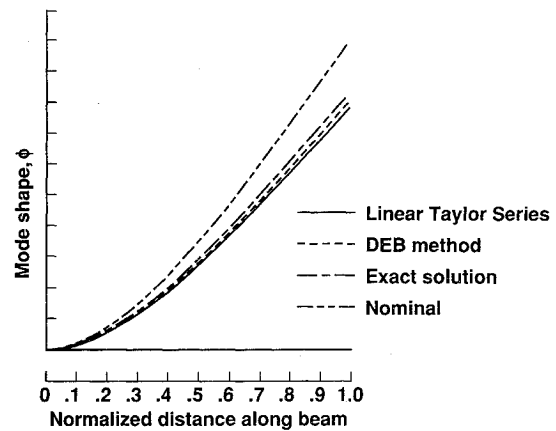

Fig. 6   Comparison of DEB and Taylor series approximations of first bending mode shape for 50% increase in beam width B (nominal values: $H = 5.0$ in.; $B = 3.75$ in.).
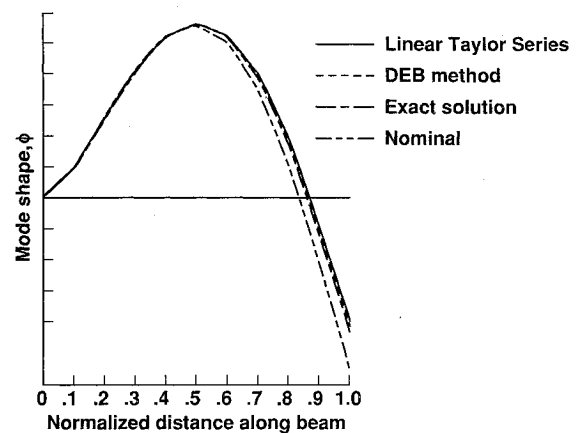

Fig. 7   Comparison of DEB and Taylor series approximations of second bending mode shape for 80% increase of tip mass M (nominal values: $B = 3.75$ in.; $H = 5.0$ in.; $M = 5.0$ lbm).

Mode Shape Approximation Results

Figure 6 shows the first bending mode shape for the nominal design, the linear Taylor series approximation, the DEB approximation, and the exact solution for a 50% increase in the beam width B. Figure 7 shows the second bending mode shape for the nominal design and both approximations and the exact shape for an 80% increase in the tip mass. Although the linear Taylor series approximation is very accurate, the DEB approximation is slightly closer to the exact curve in both figures. However, the mode shapes do not seem to be sensitive to design changes for this test problem, as indicated in Fig. 6. In some cases, the DEB method did not approximate the mode shapes as accurately as the linear Taylor series approach. Usually, these were cases in which the change in the mode shape due to perturbations in the design were small. Other test cases may provide a more challenging problem for testing these approximation methods for mode shapes and will be a topic for future investigation.

Static Displacement Approximation Results

The cantilever beam of Fig. 1 was also used to test the method for approximating static displacements. The design variables were the dimensions H and B. A distributed loading was used to represent the applied force, and the load values at each grid point of the model are given in Table 2. Figure 8 shows a typical convergence plot for the DEB method Eq. (16) for the maximum displacement. This figure suggests that the number of terms needed for a converged value is 3 or 4. For this same number of terms and the same amount of computation, the Taylor series approximation [Eq. (19)] is known to be

more accurate and would be chosen over the DEB method unless it did not converge for certain values of Δv. Results for the linear Taylor series will be included because it is a frequently used method of approximation.

Figures 9–11 show results for perturbation of the nonlinear design variable H. In Fig. 9, the maximum displacement for values of H from 3.5 in. to 8.5 in. was plotted for the DEB, linear Taylor series approximations, converged Taylor series, and the exact solutions. As H is decreased, the bending stiffness of the beam is decreased causing the beam displacement to become very large. At this end of the range it is difficult for the approximation methods to follow the curvature of the exact results. At the other end, however, the DEB method appears to follow the trend of the exact curve. The linear Taylor series approximation gives negative values for the displacement beyond 40% perturbation of H from the nominal value.

Figure 10 is a graph of the displacement of the beam for a 30% decrease in the nonlinear variable H. The figure compares the DEB and linear Taylor series approximations to the exact solution. Both approximations are very good with the DEB method slightly more accurate. A similar comparison is shown in Fig. 11. For a 50% increase in H from the nominal value, the DEB method had an error of 47% for the maximum displacement. However, the linear Taylor series approximation gives a displacement in the opposite direction. This is in accordance with the result we saw in Fig. 9.

Figures 12–14 show results for linear Taylor series and DEB displacement approximations, converged Taylor series, and exact solutions for the linear variable B. In Fig. 12, the maximum displacement for values of B from 1.875 to 5.625 in. was plotted for each approximation method and compared to the exact solution. Again, it appears to be more difficult to approximate the displacement for small beam dimensions, but the DEB method follows the exact curve, particularly for the

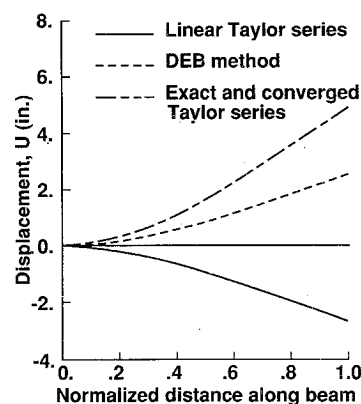Table 2   Values of applied force at the grid points of the beam model

Grid point	Force, lbf
1	0.000
2	0.025
3	0.200
4	0.675
5	1.600
6	3.125
7	5.400
8	8.575
9	12.800
10	18.225
11	25.000
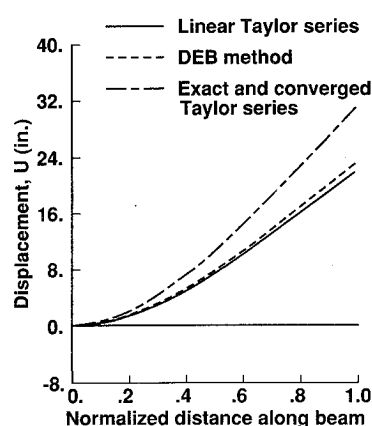
Fig. 10   Comparison of DEB and Taylor series approximations of displacement for 30% decrease in beam depth H.
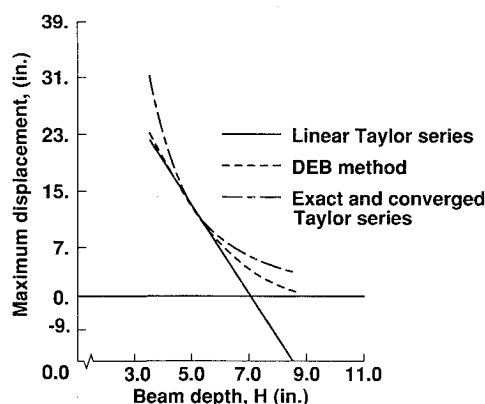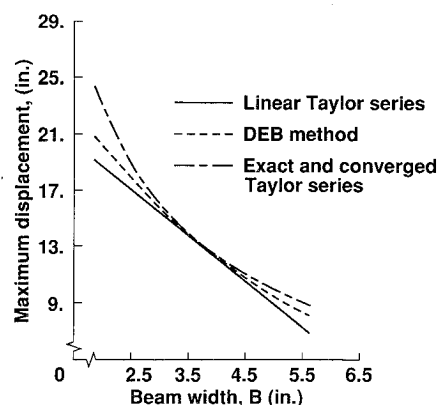

Fig. 11   Comparison of DEB and Taylor series approximations of displacement for 50% increase in beam depth H.
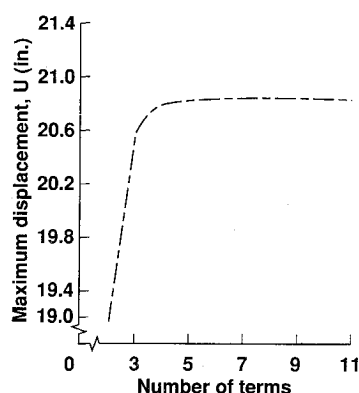
Fig. 8   Convergence plot of DEB method.

Fig. 9   Comparison of DEB and Taylor series approximations of maximum displacement vs perturbation of beam depth H.

Fig. 12   Comparison of DEB and Taylor series approximations of maximum displacement vs perturbation of beam width B.
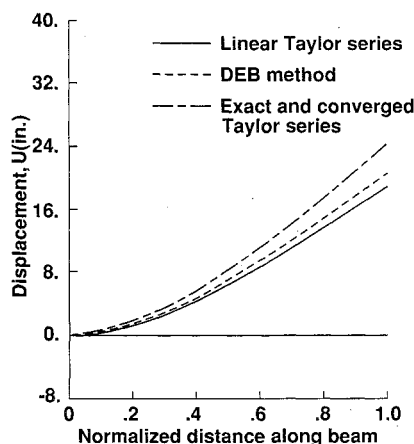
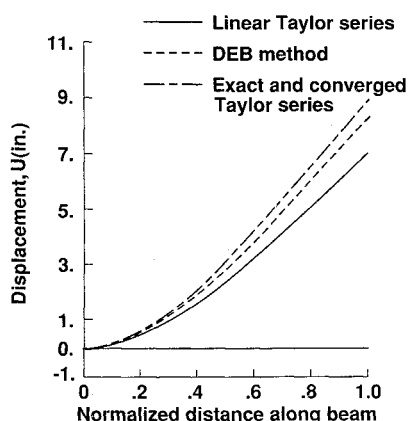Fig. 13 Comparison of DEB and Taylor series approximations of displacement for 50% decrease in beam width *B*.

Fig. 14 Comparison of DEB and Taylor series approximations of displacement for 50% increase in beam width *B*.

Fig. 15 Comparison of DEB and Taylor series approximations of maximum stress vs perturbation of beam width *B*.

Fig. 16 Actual values of coefficients assumed constant in DEB frequency approximation for simultaneous perturbation of tip mass, bending inertia, and cross-sectional area.

Fig. 17 Actual values of coefficients assumed constant in DEB mode shape approximation for mass perturbation.

larger values of *B*. Figure 13 shows a plot of the approximate displacements and the exact solution for a 50% decrease in *B*. Both approximations are good but the DEB curve is closer to the exact. Figure 14 shows the result for a 50% increase in the design variable. It shows that the DEB approximation is within 8% of the exact solution compared to the linear Taylor series error of 21%.

Additional Approximations

An additional application of the DEB approximation is in estimating system response quantities that depend on approximated quantities. Inserting the approximate displacement field into the stress-displacement equations to obtain approximate stress field is an example of this. Figure 15 shows a graph of maximum normal bending stress for perturbed values of *B*. The stresses were obtained using the approximate values of *U* from the DEB method and the linear Taylor series approach and are compared to the converged Taylor series and exact solutions. The approximate stress field shown in Fig. 15 is very accurate. This was a particularly good result in view of the fact that the stresses are in effect proportional to the second derivatives of the approximate displacements.

Validity of Assumptions Made in DEB Approximation Method

Recall that the methods developed in this paper were based on the assumption that certain variables were assumed constant, for example, *a* and *b* in Eqs. (5) and (26). In order to assess the validity of these assumptions, the actual values of the coefficients a_θ and b_θ were plotted for values of the tip mass, bending inertia, and cross-sectional area perturbed
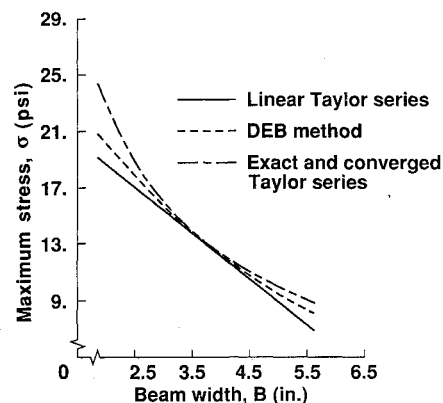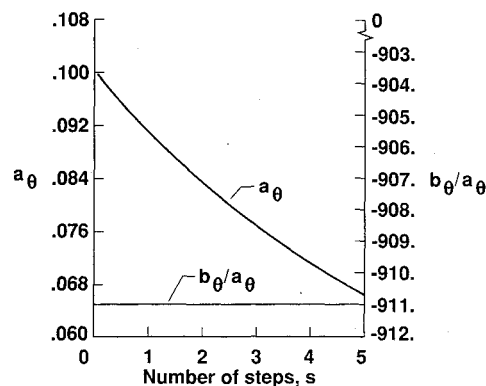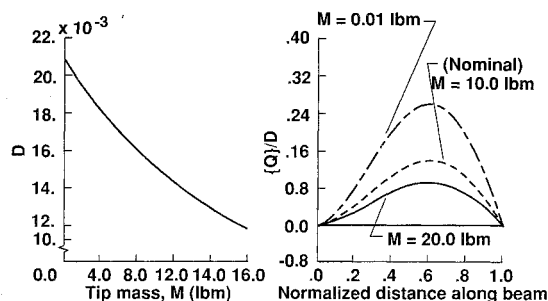
simultaneously. Figure 16 shows a graph of a_θ and b_θ/a_θ vs the number of 10% step increases in the design variables. The lower curve shows that b_θ/a_θ is essentially constant. The upper curve, representing the coefficient a_θ, shows a deviation of up to 19% from the nominal value when the design variables are changed by up to 50%. Assuming a_θ to be constant was not detrimental to the approximation results. As shown in Fig. 4, the DEB method was able to obtain very good approximations for the full range of the perturbed design variables. Figure 17 shows a graph of the coefficients that appear in the mode shape approximation of Eq. (10). The plot of *D* vs perturbed values of tip mass show ± 20% deviations in *D* based on up to 100% perturbation in the tip mass. The second plot shows the vector *Q/D* for the nominal tip mass and for ± 100% perturbations from the nominal mass. These plots show that the vector varied considerably; however, as shown in Fig. 6, this did not seem to have a large effect on the approximate mode shapes obtained with the DEB method.

Concluding Remarks

This paper described a method that is based on interpreting sensitivity equations of structural response as differential equations that may be solved for closed-form approximations. The method was developed for approximating vibration frequencies, mode shapes, and static displacements. The test cases used to evaluate the approximations involved perturbations of the height, width, cross-sectional area, tip mass, and bending inertia of a cantilever box beam. The results were compared to the linear Taylor series approximations and exact solutions. The results showed that the new method is very accurate and in most cases more accurate than the linear Taylor series approximations. The method is applicable to both linear and nonlinear design variables and to simultaneous perturbation of several design variables. Also, the approximated response may be used to approximate behavior derived from the response, for example, approximating stresses from an approximate displacement field. To date, the method has been developed and demonstrated for frequencies, mode shapes, static displacements, and stresses derived from the displacement. In principle, the method is applicable to approximating any quantity for which an analytical sensitivity formula is available.

Acknowledgment

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